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RESEARCH

Design and Implementation of a Learning Task for the Study of Sine and Cosine Functions

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ABSTRACT

PURPOSE: This research aims to design and implement a Mathematical Learning Task (MLT) to teach trigonometric functions, specifically sine and cosine, to first-year high school students. It addresses difficulties in teaching and learning these concepts, particularly due to the dichotomous tradition of teaching them either as ratios in right-angled triangles or as functions of a real variable.

DESIGN/METHODOLOGY/APPROACH: The study involved the development of a learning task that incorporated a physical manipulative, followed by a comparison of traditional pencil-and-paper methods with digital tools. The task focused on promoting activities such as pattern identification, conjecture formulation, and result communication, all essential for deepening students' understanding of mathematical concepts.

FINDINGS: The results showed an increase in student interest and a better understanding of key ideas and concepts related to trigonometric functions. Additionally, an evaluation rubric was developed based on the cognitive demand of the tasks; this helped assess the depth of student engagement and comprehension.

RESEARCH LIMITATIONS/IMPLICATIONS: While the findings are promising, further studies are needed to explore the long-term impact of such tasks on students' overall understanding of trigonometry and their ability to apply these concepts in different contexts.

ORIGINALITY/VALUE OF THE PAPER: This study contributes to the teaching of trigonometry by highlighting an innovative approach that blends traditional and digital methods, offering a more engaging and comprehensive way to teach complex mathematical functions to high school students.

PRACTICAL IMPLICATIONS: The findings suggest that integrating physical manipulatives with digital tools can enhance student learning, providing teachers with a new approach to improving students' understanding of trigonometric functions.

KEYWORDS: *Mathematical Learning Tasks; Sine and Cosine Functions; Cognitive Demand; Digital Tools*

INTRODUCTION

Trigonometry, originating from ancient Western traditions, focuses on the relationships between the sides and angles of triangles. The term, derived from Greek meaning "triangle measurement", has been integral to the development of mathematics. An early 20th century discovery, the Plimpton tablet from around 1800BC, highlights its ancient relevance, presenting numerical information related to right-angled triangles and Pythagorean triads (Rondero *et al.*, 2018). Over time, trigonometry has played a pivotal role in scientific advancements, notably with Joseph Fourier's application of trigonometric functions in wave theory during the 17th century (Hallinan and Sandford, 2019).

In modern education, trigonometry is introduced in secondary schools, focusing on solving right-angled triangles through memorisation of trigonometric ratios. However, this method often lacks meaningful connection to real-world contexts, limiting students' engagement and understanding (Kendal and Stacey, 1998). Textbooks may present inaccessible problems, such as measuring river widths or lighthouse heights, which many students find disconnected from their realities (Andrade *et al.*, 2020). Therefore, curriculum reforms emphasise solving relevant, community-related problems (Ministry of Public Education, 2017).

At the higher education level, understanding trigonometric functions is essential, particularly in fields such as engineering (Torres-Corrales and Montiel-Espinosa, 2021). Teaching strategies should evolve to incorporate diverse pedagogical tools, manipulatives, digital resources, and problem-solving exercises, to enhance students' learning experiences. Such approaches can foster deeper mathematical thinking, helping students' transition from understanding basic trigonometric ratios to more complex functions (Cañadas and Castro, 2022). The strategies that are proposed in the upper secondary level study programmes require a work of reflection, analysis, and recognition of mathematical content to address learning tasks and the solution of real problems, beyond the direct application of a mathematical resource.

THEORETICAL REFERENCES

On Learning Trigonometric Functions

During the teaching of this topic, various difficulties have been identified, as noted by Franchi and Hernández de Rincón (2004), Godino *et al.* (2003), and Zubieta (2018). The literature review reveals that the conceptions of sine and cosine functions originated as ratios of sides in right-angled triangles. However, in more recent mathematical history, they have been expanded to include functions, linking an independent variable to a dependent one. This broader view also incorporates concepts such as domain, range, limits, and continuity.

Altman and Kidron (2016) highlight difficulties in teaching trigonometric functions, particularly the focus on their interpretation solely as ratios between two quantities, as in right-angled triangles. This emphasis can hinder understanding of these functions as circular or real variable functions. Furthermore, students often struggle to move beyond the right-angled triangle context, and this limits their ability to fully grasp these functions as independent concepts.

Torres-Corrales and Montiel-Espinosa (2021) argue that students, especially at the upper secondary and first-year university levels, need to reconceptualise trigonometric functions. Additional challenges include using degrees for angle measurement in right-angled triangles versus radians in circular trigonometry and polar co-ordinates. Also, the right-angled triangle context promotes a static view of angles, while the trigonometric circle offers a dynamic perspective.

These issues are part of a broader dichotomy of conceptions, where triangular and circular trigonometry developed as separate epistemic traditions (Bressoud, 2010; Thompson, 2008). Each tradition's distinct historical development contributes to the distance between these concepts.

About the Design of Learning Tasks

This work focuses on the transition from trigonometric ratios to trigonometric functions through the unit circle, specifically for high school students. It highlights the importance of the teacher's role in addressing potential challenges, particularly through well-designed learning tasks (MLT, Torres-Rodriguez *et al.*, 2022; Stein *et al.*, 2007) and the Cognitive Demand of Learning Tasks (Stein and Smith, 1998). Key factors in task design include promoting exploration, connections, and understanding of sine and cosine functions.

While traditional methods emphasise paper-and-pencil problem-solving, the integration of digital tools such as GeoGebra offers a dynamic way to visualise concepts and facilitate the transition from ratios to functions (Gutiérrez, 2016). This study aims to design, implement, and evaluate a learning task that integrates these principles to enhance students' understanding of trigonometric functions, using problem-solving strategies and digital tools (Barrera-Mora *et al.*, 2021).

METHODOLOGY

The work was developed from a qualitative persepctive and the information was analysed from data collection instruments such as video recording transcripts and the collection of written records of the students. These two sources provided us with valuable information for the research.

For the elaboration of the written tests, the design of a sequence of activities was considered with the aim of structuring a Mathematical Learning Task (MLT). This would help to solve some of the difficulties previously identified in the understanding of the mathematical concepts involved in the topic in question.

Study Subjects

The learning task was applied to high school students belonging to the Community teleworking study programme in a rural community in the municipality of Huasca de Ocampo, State of Hidalgo, in Mexico. A total of 11 second-semester students participated, and all students were between the ages of 15 and 17 at the time of the study. The recordings were transcribed in digital format, as were the written tests, and each student was assigned a letter to protect their personal identity.

The Learning Task

A learning task was designed that included three didactic sequences, described as a starting activity with a duration of four hours, a preliminary activity with a total of four hours, and finally the main activity with a duration of five hours. The proposal was designed based on the analysis of previous works by Maknun *et al.* (2019), Altman and Kidron (2016), and Hertel and Cullen (2011) that offer key elements to consolidate the design. The general proposal is shown in Figure 1.

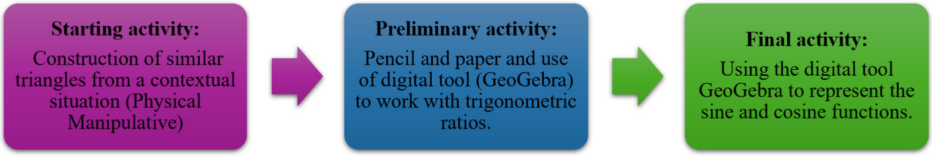


Figure 1: Proposed Learning Task, Composed of Three Didactic Sequences

Source: Constructed by authors

The so-called starting activity was divided into two sessions. Its purpose was to introduce the topic by constructing similar triangles, from a contextual situation, through the use of a physical manipulable, as well as with the use of GeoGebra software (see Figure 2).

The central idea was to achieve a geometric representation of similar right-angled triangles in the context of the physical manipulables, and from this obtain values by measuring the distance in each of the traced segments. Subsequently, with the data obtained, students would be able to establish the two trigonometric ratios of interest in this study by filling out a written record. An important thing that was considered during the design stage of this instrument was that it should start with a different reference angle of 45°.

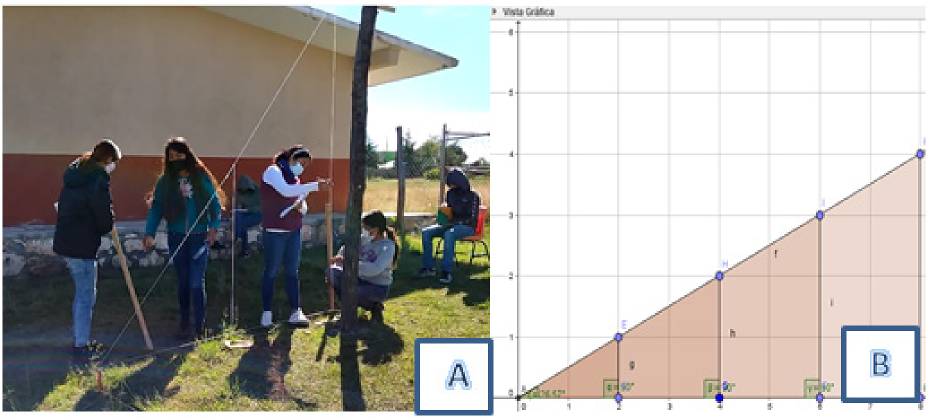


Figure 2: The Two Sessions of the Starting Activity

Source: Constructed by authors

In the first session, a geometric representation designed in the GeoGebra software, which is observed in the diagram of Figure 2a, was used to guide the students in the use of the manual tool (Figure 2b). For this first approach, a reference angle with a value of 60° was considered; this value was proposed taking into account that the magnitude of the segments would be adequate to obtain data that would serve for subsequent activity. The use of GeoGebra is justified because it allows the visualisation of the similarity between right-angled triangles to be seen more easily, and allows the determination of the magnitude between legs, using the tools offered by the software.

In the second session, a physical device was collectively built using thread and pieces of wood, forming a pair of similar triangles. This was related to the problem as it contained what is known in construction and masonry as a “plumb line”; this uses gravity to establish the verticality of a line, as well as its perpendicularity with respect to the ground. In this second session, the students were given worksheetson which they recorded the ideas inferred from the previous activity; with the data obtained from the measurements they established the trigonometric relationships between sine and cosine with the angle of reference. Finally, the students were asked to draw right-angled triangles with reference angles of 15°, 30° and 45° on paper with the use of geometric play; the trigonometric ratios were also requested. The teacher guided the students during the process with a technical sheet.

The second activity focused on sine and cosine trigonometric ratios: it included two 2-hour sessions. In the first session, students used pencil and paper, completing a worksheet with geometric diagrams (Figures 3a and 3b) and a written-tabular recording. The teacher guided the activity using a technical sheet.

For the second session, the GeoGebra digital tool was used. The students had previously received instructions on the use of the tool meaning they knew certain elements of the software. For this activity, a figure described as a “right triangle inscribed in a circle of radius one” was constructed (see Figure 3c). This scheme was used as a reference to analytically calculate the value of the trigonometric sine and cosine ratios of 45°. The duration of this activity was two hours.

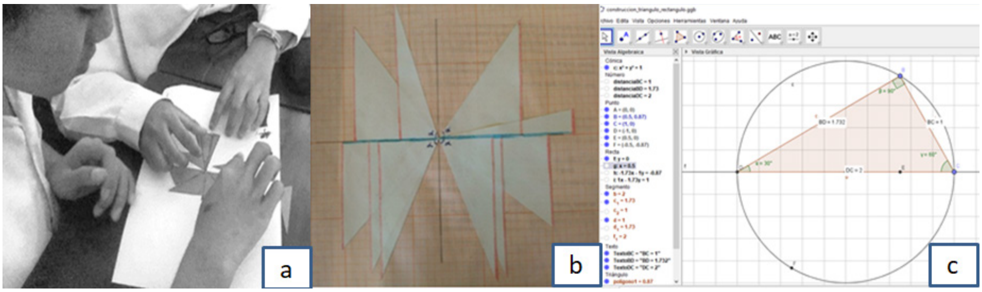


Figure 3: Figures Corresponding to the Sessions that made up the Preliminary Activity, First Session (3a and 3b), Second Session (3c)
Source: Constructed by authors

There were two sessions in the final activity. The first session was mainly dedicated to the use of the GeoGebra software by the students. The students undertook a geometric construction of a triangle inscribed in a unitary circle (Figure 4a), the concept of trigonometric ratios for notable angles was worked on, in addition to considering the value of 0° and 90°. In addition to this, the idea of a right-angled triangle in a unitary circle was extended to a right angle inscribed in a circle of radius two (Figure 4b); with this, the work previously carried out was consolidated. In addition to these geometric constructions, the students answered in a written form in which they detailed the value of the trigonometric ratios for both representations.

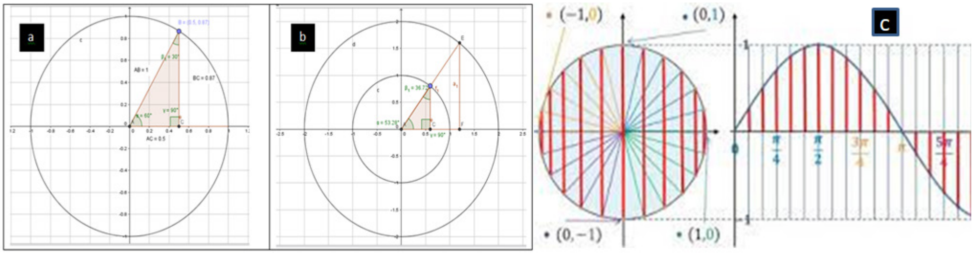


Figure 4: Sessions that made up the Final Activity: Figures 4a and 4b Correspond to the First Session and Figure 4c to the Second

Source: Constructed by authors

Finally, the last session was dedicated to developing and understanding the concept of trigonometric function. The students worked on a written record with the intention of bringing them closer to the meaning through graphic and tabular representation. The teachers wanted the students to represent the trigonometric functions sine and cosine graphically, taking the special care required when representing the domain of the function in terms of real numbers. For this reason, the conversion of degrees (sexagesimal representation) to radians was emphasised, thus considering a periodic function (see Figure 4c).

Evaluation of the Main Activity

For the main activity, whose purpose was to use the software to represent the sine and cosine functions in their circular nature and relate them to the notion as real variable functions, the use of an evaluation framework based on the extended classification of Benedicto *et al.* (2015) was considered. Table 1 shows a simplified table with the categories and criteria considered in this classification.

Table 1: Criteria for Assessing Levels of Cognitive Demand

Level of Cognitive Demand (DC)	*Standard*	*Criteria*
LOW	Memorisation	1.1 to 1.6
LOW-MEDIUM	Offline algorithms	2.1 to 2.6
MEDIUM-HIGH	Algorithms with connections	3.1 to 3.6
HIGH	Do Math	4.1 to 4.6

Source: Benedicto *et al.*, 2015

This evaluation stage allowed us to identify to what extent students were able to make this transition from the notion of trigonometric ratios to trigonometric functions; this was the stage we wanted to reach in the final part of the learning task.

RESULTS

Startup Activity

The purpose of this activity was for the students to build similar triangles from a physical manipulative, support themselves by doing an analysis with pencil and paper, and later using GeoGebra to represent the same situation. In the first session, it was difficult for them to project similar right-angled triangles, using the physical manipulative. However, tutors helped them by manipulating the length of the segments and angles so they were able to understand the representations sought. They could, therefore, begin to understand that the trigonometric ratios of sine and cosine can be estimated for any measurement of angle, and of the legs, trying to break previous ideas about fixed or static values. Figure 5 shows one student's answers in a collected written production that intended to calculate the value of the trigonometric sine and cosine ratios for different configurations of sides and angles.

Apartado I. Hallar el valor de las razones trigonométricas seno y coseno para el ángulo de referencia de 60°, para llenar la siguiente tabla debes considerar los valores obtenidos en la representación gráfica que se realizó en la actividad "triángulos rectángulos semejantes"

Valor del ángulo	Cateto menor	Cateto mayor	Hipotenusa	Razón trigonométrica seno	Razón trigonométrica coseno
60°	1m	1.17m	1.59m	seno = 1.17m / 1.59m Seno 0.73	Coseno = 1 / 1.59 coseno 0.62
60°	2m	2.37m	3.20m	seno 2.37m / 3.20m Seno = 0.74	Coseno = 2 / 2.37m coseno = 0.62
60°	3m	3.51m	4.77m	seno = 3.51m / 4.77m Seno = 0.73m	3 / 4.77 coseno = 0.62

Figure 5: Calculation of Values for Trigonometric Ratios for Different Configurations of Angles and Sides
Source: Constructed by authors

In the second session, the use of GeoGebra was able to help the students overcome the difficulties that were observed in the session with the physical manipulative. During this session, the teacher guided the constructions made in the software, in addition to the use of the available tools, values of angles, sides, and with them the values of the trigonometric ratios.

Preliminary Activity

In this second activity, materials and strokes made with millimetric sheets of paper were used. The purpose was to move towards the idea of trigonometric functions of sine and cosine, through the identification of the geometric place that is formed when right-angled triangles of different configurations of their legs are drawn, but with the same length of hypotenuse (see Figure 6).

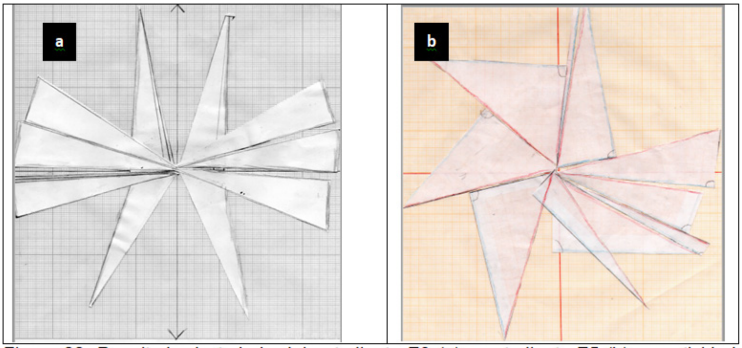


Figure 6: Work Developed, Student E3 (a) and Right-Angled Triangles with Hypotenuse Equal to 10cm, Student E5 (b)
Source: Constructed by authors

The first idea that the students discussed was that the geometric place in question was about irregular polygons. It was necessary to have the teacher's guidance and to work on a greater number of different configurations of triangles so that, finally, several of the students identified that the figure that was formed was a circle, whose radius coincided with the length of the hypotenuses of the triangles considered. A situation that arose had to do with confusing on some occasions one of the legs with the hypotenuse. In the second session of this activity, we worked again with GeoGebra to solve these difficulties; to do this, it was necessary to make several constructions of triangles inscribed in circles.

Main Activity

The evaluation criteria depended on the type of understanding that the students acquired, according to the quality of their answers. With this information, it was identified at which level(s) achievements could be corroborated. In general, only the so-called low/medium and medium/high levels of the classification of Benedicto *et al.* (2015) were reached. The characteristics for the aforementioned levels are presented in Table 2 and some evidence that agrees with these results is shared.

In general, the use of the digital tool made it possible to better manipulate the geometric figure, being able to draw more figures in the same Cartesian plane, as well as being able to perform the calculations of angles and segments more accurately (levels 3.1 and 3.4 in Table 2).

Table 2: Levels of Cognitive Demand and Criteria, Task Assessment

LOW-MEDIUM	*MEDIUM-HIGH*
(2.1) They are algorithmic. They expressly indicate which algorithm to use or it is evident from the context.	(3.1) They guide the student to use algorithms with the aim of having a deeper understanding of mathematical concepts and ideas.
(2.4) Focused on obtaining correct answers but not on developing mathematical understanding.	(3.4) Its successful resolution requires some cognitive effort. General algorithms can be used, but when applying them, you have to pay attention to the structure of the pattern.
(2.5) Explanations that focus solely on describing the algorithm used.	(3.5) Students need to consciously consider conceptual ideas underlying algorithms in order to successfully solve the question.
(2.6) Multiple representations can be used in solving them (arithmetic, visual diagrams, manipulative materials, etc.).	(3.6) Explanations that refer to the underlying relationships using concrete cases (particular positions in the series).

Source: Benedicto *et al.*, 2015

Some of the students managed to understand the transition from trigonometric ratios to their representation in the unitary circle, although the comprehension was not entirely complete for angles greater than 360°. In circular trigonometry, angles greater than 360° represent multiple rotations around the circle. This is essential in understanding periodicity and the extension of the sine and cosine functions beyond a single rotation (Weber, 2005).

E1: Because its sides are proportional, and because we handle the same number of decimals, and the more decimals we have throws us to a value closer to the real one.

E2: As long as the triangles have similarities and their relationships are the same, it does not matter if there are 10, 20, 30, 100, or 1,000 triangles; if they are similar they will give the same value of angles (see criterion 2.6 in Table 2).

To formalise the ideas, the group worked on a written record stating the feasibility of obtaining data with digital technology so that students could achieve results closer to the real value of trigonometric ratios for right-angled triangles inscribed in circumferences of different radii (see Figure 7).

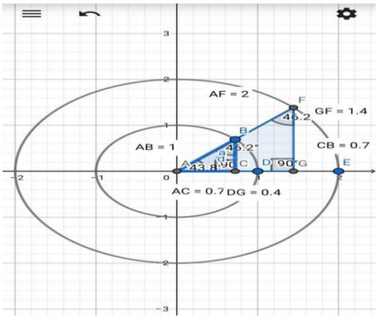


Figure 7: Use of the Digital Tool on a Mobile Device, Student E11

Source: Constructed by authors

Most students reach a level where they do algorithmic procedures with a certain level of understanding (see criterion 3.1 in Table 2). Figure 8 shows an example of the written records made by a student with the data obtained from GeoGebra.

1. Completa la siguiente tabla con datos obtenidos empleando la herramienta GeoGebra

$n = \frac{C.O}{h}$

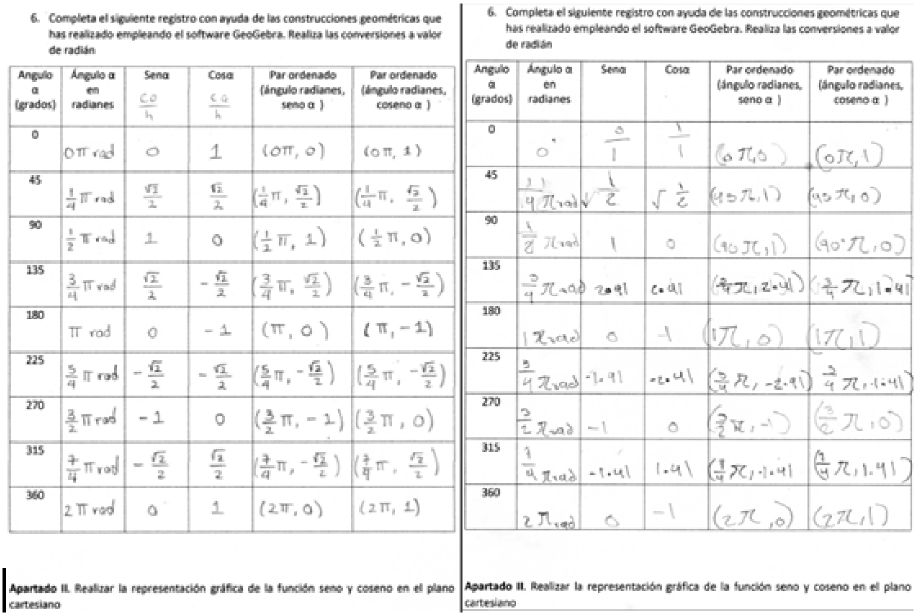
$s = \frac{C.a}{h}$

	Triángulo rectángulo inscrito en circunferencia con radio igual a 1		Triángulo rectángulo inscrito en circunferencia con radio igual 2	
Medida del ángulo (grados)	Razón trigonométrica seno	Razón trigonométrica coseno	Razón trigonométrica seno	Razón trigonométrica coseno
15	$Sen = \frac{0.26}{1}$ $Sen = 0.26$	$Cos = \frac{0.97}{1}$ $Cos = 0.97$	$Sen = \frac{0.52}{2}$ $Sen = 0.26$	$Cos = \frac{1.94}{2}$ $Co = 0.97$
30	$Sen = \frac{0.5}{1}$ $Sen = 0.5$	$Cos = \frac{0.866}{1}$ $Cos = 0.866$	$Sen = \frac{1}{2}$ $Sen = 0.5$	$Cos = \frac{1.732}{2}$ $Cos = 0.866$
45	$Sen = \frac{0.707}{1}$ $Sen = 0.707$	$Cos = \frac{0.707}{1}$ $Cos = 0.707$	$Sen = \frac{1.414}{2}$ $Sen = 0.707$	$Cos = \frac{1.414}{2}$ $Cos = 0.707$
60	$Sen = \frac{0.87}{1}$ $Sen = 0.87$	$Cos = \frac{0.5}{1}$ $Cos = 0.5$	$Sen = \frac{1.73}{2}$ $Sen = 0.865$	$Cos = \frac{1}{2}$ $Cos = 0.5$
0	$Sen = \frac{0}{1}$ $Sen = 0$	$Cos = \frac{1}{1}$ $Cos = 1$	$Sen = \frac{0}{2}$ $Sen = 0$	$Cos = \frac{2}{2}$ $Cos = 1$
90	$Sen = \frac{1}{1}$ $Sen = 1$	$Cos = \frac{0}{1}$ $Cos = 0$	$Sen = \frac{1}{2}$ $Cos = 1$	$Sen = \frac{0}{2}$ $Cos = 0$

Figure 8: Approximation to Trigonometric Ratios with Data Obtained from GeoGebra

Source: Constructed by authors

For the second session of the main activity, the objective was to achieve a greater understanding of the characteristics of the sine and cosine as functions, through the construction in GeoGebra, based on the tabular and graphical records. Through class work, the teacher addressed the concepts of periodicity, range, and domain in real numbers to express in terms of real values, as well as the use of the values of angles in radians. In this session, it was not possible for all students to understand these meanings associated with the functions of sine and cosine. The information was also collected through a written record of the data provided by the software (see Figure 9).



6. Completa el siguiente registro con ayuda de las construcciones geométricas que has realizado empleando el software GeoGebra. Realiza las conversiones a valor de radián

Angulo α (grados)	Ángulo α en radianes	Sena $\frac{co}{h}$	Cosa $\frac{ca}{h}$	Par ordenado (ángulo radianes, seno α)	Par ordenado (ángulo radianes, coseno α)
0	0π rad	0	1	$(0\pi, 0)$	$(0\pi, 1)$
45	$\frac{1}{4}\pi$ rad	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$(\frac{1}{4}\pi, \frac{\sqrt{2}}{2})$	$(\frac{1}{4}\pi, \frac{\sqrt{2}}{2})$
90	$\frac{1}{2}\pi$ rad	1	0	$(\frac{1}{2}\pi, 1)$	$(\frac{1}{2}\pi, 0)$
135	$\frac{3}{4}\pi$ rad	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$(\frac{3}{4}\pi, \frac{\sqrt{2}}{2})$	$(\frac{3}{4}\pi, -\frac{\sqrt{2}}{2})$
180	π rad	0	-1	$(\pi, 0)$	$(\pi, -1)$
225	$\frac{5}{4}\pi$ rad	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$(\frac{5}{4}\pi, -\frac{\sqrt{2}}{2})$	$(\frac{5}{4}\pi, -\frac{\sqrt{2}}{2})$
270	$\frac{3}{2}\pi$ rad	-1	0	$(\frac{3}{2}\pi, -1)$	$(\frac{3}{2}\pi, 0)$
315	$\frac{7}{4}\pi$ rad	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$(\frac{7}{4}\pi, -\frac{\sqrt{2}}{2})$	$(\frac{7}{4}\pi, \frac{\sqrt{2}}{2})$
360	2π rad	0	1	$(2\pi, 0)$	$(2\pi, 1)$

Apartado II. Realizar la representación gráfica de la función seno y coseno en el plano cartesiano

6. Completa el siguiente registro con ayuda de las construcciones geométricas que has realizado empleando el software GeoGebra. Realiza las conversiones a valor de radián

Angulo α (grados)	Ángulo α en radianes	Sena	Cosa	Par ordenado (ángulo radianes, seno α)	Par ordenado (ángulo radianes, coseno α)
0	0	$\frac{0}{1}$	$\frac{1}{1}$	$(0\pi, 0)$	$(0\pi, 1)$
45	$\frac{1}{4}\pi$ rad	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	$(45\pi, 1)$	$(45\pi, 0)$
90	$\frac{1}{2}\pi$ rad	1	0	$(90\pi, 1)$	$(90°\pi, 0)$
135	$\frac{3}{4}\pi$ rad	2.41	0.41	$(\frac{3}{4}\pi, 2.41)$	$(\frac{3}{4}\pi, 1.41$
180	1π rad	0	-1	$(1\pi, 0)$	$(1\pi, 1)$
225	$\frac{5}{4}\pi$ rad	-1.91	-2.41	$(\frac{5}{4}\pi, -2.91)$	$\frac{5}{4}\pi, -1.41)$
270	$\frac{3}{2}\pi$ rad	-1	0	$(\frac{3}{2}\pi, -1)$	$(\frac{3}{2}\pi, 0)$
315	$\frac{7}{4}\pi$ rad	-1.41	1.41	$(\frac{7}{4}\pi, -1.41$	$(\frac{7}{4}\pi, 1.41)$
360	2π rad	0	-1	$(2\pi, 0)$	$(2\pi, 1)$

Apartado II. Realizar la representación gráfica de la función seno y coseno en el plano cartesiano

Figure 9: Trigonometric Functions, Written Records of E2 and E5 Students

Source: Constructed by authors

CONCLUSIONS

Around the Design of the Learning Task

The conceptual reference of the design of a mathematical learning task and the problem-solving approach provided characteristics to the tasks, such as the inquisitive process that the teacher used to guide the activities, the interaction that was encouraged among the students, and the heuristics that the teacher used for the activities. To a large extent, all of them involved the participation of the students, as shown in the three areas marked as the initial, preliminary, and main processes. The problem-solving approach was the central axis for the design of the MLT; this orientation allowed the focus to be on activities where students used mathematical resources learned in the classroom, such as procedures or algorithms; these were also used as a strategy to formulate conjectures from observations such as patterns in geometric figures and with this hold a group discussion about the main findings obtained. The three sequential processes were intended to contribute to a better understanding of the problem-solving process for the design of the MLT.

The elements that are present in the framework of cognitive demand (Stein and Smith, 1998), and that were considered in the design of the learning task, are the support of manipulatives and visual diagrams. Multiple representations were considered as referents in sessions one, two, three and four. In this order of ideas, the activities were designed with the use of manipulatives in the real context, where the students relied on manual tools to make a geometric construction. The use of visual diagrams was addressed when students worked with the manipulation of school tools, in the context of pencil and paper. Finally, the visual diagrams were designed for the activities whose design encouraged the use of the GeoGebra tool.

Continuing with the characteristics of cognitive demand, in addition to the development of mathematical thinking, it was also intended to achieve a robust connection of concepts and meanings with the discussion of ideas. This was done for sessions four, five, and six that were designed so that students could retrieve prior knowledge and discuss ideas in a group so that they would eventually connect the content of unit-circle trigonometric ratios and the transition to trigonometric functions.

Around the Results Obtained

With reference to the conceptual construct of cognitive demand, it is considered that most students reach the medium-high level of demand. The main characteristic of this level is called the representation of the solution, consisting of the resolution connecting various representations. These are shown in various ways, and students use those that lead them to more abstract reasoning according to the criteria of the table proposed by Benedicto *et al.* (2015). It shows that the students in sequence number one were generating ideas when dealing with similar triangles and that they also realised that, invariably, when moving an acute angle located on the basis of the geometric construction, a certain number of right-angled triangles with proportional sides were generated. Following up on these ideas, the students realised that the acute angle must be greater than 0° but less than 90° in order to continue maintaining the criteria they knew about the geometric construction of right-angled triangles.

Discussing the results obtained during the implementation stage, around the theoretical reference of the use of digital tools, we can highlight the use of the GeoGebra software for the sequence of activities. Number two allowed them to draw a right-angled triangle inscribed in a circle of radius two, formalising the idea that, regardless of the value of the radius, the trigonometric ratios are conserved; the students performed arithmetic operations to calculate the value of the sine and cosine ratios. Another situation that allowed linking the ideas of trigonometric ratios was to build similar right-angled triangles in circles of different radii, using the GeoGebra software. This situation saw the students reach the medium-high level of demand; this category specifies that successful resolution requires some cognitive effort. It also considers that the student can use general algorithms, but when applying them, attention must be paid to the structure of the pattern (Benedicto *et al.*, 2015).

Specifically, the sequence of activities designed for session six meant that the student could move towards the understanding of trigonometric function, relying on the representation of the unitary circle. In this way, it was intended to break with the two separate ideas or conceptions identified by Bressoud (2010).

During the session described above, the students managed to understand the concept of a periodic function, only in a unit circle, performing the representation in GeoGebra. The transition that was achieved towards the idea of function with variables in the co-ordinate plane of real numbers was not fully understood; this is because the students only made the conversion of values in sexagesimal degrees to values such as real numbers (radians) but less than 2π.

One of the difficulties faced by most of the students was in the representation of the sine and cosine ratios for angles greater than 90°, as they did not achieve a graphic representation strategy, such as similar right-angled triangles embodied in the different quadrants. They did not succeed in fully moving towards the notion of sine function as an expression of a real variable.

Based on the results obtained, the traits classified by Benedicto *et al.* (2015) were observed in some students in relation to the construct of cognitive demand. Specifically, it was possible to corroborate that the transition from the medium-high to the low-medium levels was largely due to the management they had of trigonometric ratios, and some shortcomings in understanding trigonometric functions since the students showed evidence of processes that are described in the criteria of these two categories.

The objective for the evaluation of MLT was partially fulfilled: a possible argument is that some students had difficulty formulating ideas that would help them transition to the notion of trigonometric function. For this reason, the categorisation of performance as a cognitive demand was sought, considering an aspect that would achieve the identification of the achievements attained.

The evaluation process carried out may indicate the convenience of giving continuity to subsequent work that focused on structuring a more systematic evaluation scheme. As a result, it could influence improvements in the design and implementation of the different sequences that make up the task.

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